1 What is calculus?

The first key idea of calculus is "instantaneous rate of change." We want to understand and compute the instantaneous rates of change of quantities that may arise in physics, science or engineering, or other applications. The second key idea of calculus is that if we know the instantaneous rate of change of some quantity at each moment during an extended period of time, then we can compute the total change of that quantity during that period of time, because "the total change is the sum of all the little changes."

Unfortunately the instantaneous rate of change of a quantity is called the "derivative" of that quantity in calculus. This undescriptive name makes a simple idea sound more arcane and difficult than it really is.

Students of algebra understand the idea of "average rate of change" over an extended period of time.

$$\text{Avg. rate of change} = \frac{\text{change in quantity}}{\text{time elapsed}}.$$

What's new in calculus is the idea of a rate of change at a *single instant* in time. When you drive your car you may see the speedometer steadily increase from 0 km/h to 100 km/h. Along the way, there was a single instant when you were going 75 km/h.

What do we mean by the instantaneous rate of change of a quantity, and how could we compute it? The quantity could be the distance an object like a car has traveled along a straight line, for example, or the temperature of an object as measured by a thermometer. Imagine an *extremely short* period of time which includes the particular moment in time we are interested in. Compute the average rate of change of our quantity during this extremely short period of time. The number we get will be approximately equal to the instantaneous rate of change we desire.

$$\text{Instantaneous rate of change} \approx \frac{\text{little change}}{\text{short time}}.$$

If we do this again, but now using an *even shorter* period of time, the average rate of change we compute may be even closer to the instantaneous rate of change. As the duration of the period of time approaches 0, the average rate of change approaches the instantaneous rate of change. In other words, the instantaneous rate of change is just the limit of the average rate of change, as the duration of the period of time we use to compute the average approaches 0.

We see that in defining and computing the instantaneous rate of change of a quantity, the idea of a "limit" has arisen.

Now for the next main idea of calculus. Suppose we know the instantaneous rate of change of some quantity at each moment during an extended period of time, and we want to figure out the total change of the quantity during this period of time. For example, we may know the instantaneous velocity of a car during the past half hour (perhaps because we've been watching the speedometer), and we want to know how far the car traveled during that half hour. We can achieve this as follows. Break up the extended period of time into a bunch of extremely short periods of time, and compute how much the quantity changed during each of these short periods of time. The total change we desire is just the sum of all these little changes.

Each little change is approximately equal to the instantaneous rate of change at any moment (chosen arbitrarily) within the corresponding short period of time, multiplied by the duration of this short period of time. This is just a restatement of the definition of instantaneous rate of change, which was given above. Instead of saying "instantaneous rate of change $\approx \frac{\text{little change}}{\text{short time}}$", we are now saying "little change $\approx$ instantaneous rate of change $\times$ short time."

The total change is obtained by adding up all the little changes. But we only have an approximation to each of the little changes, so we only have an approximation to the total change. By repeating this whole process, but breaking up the extended period of time into a whole bunch of *even shorter* periods of time, we can get a better approximation to the total change. If we do this again and again, then in the limit as the duration of each of our sub-periods of time approaches 0, the approximation we compute will approach the true total change we desire.

This process, in which the total change is obtained as a limit of sums of tiny changes during tiny intervals of time, is given the unfortunate name "integration", another undescriptive name that makes an idea sound harder than it really is.

The fact that the total change of a quantity can be computed from the instantaneous rate of change of that quantity in this manner–in other words, that the total change can be obtained by "integrating" the "derivative"–is called the Fundamental Theorem of Calculus.